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INFORMATIVENESS OF PARALLEL KALMAN FILTERS

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Abstract-This article considers the informativeness of parallel Kalman filters. Expressions are derived for determination of the amount of information obtained by additional measurements with a reserved measurement channel during processing. The theorems asserting that there is an increase in the informativeness of Kalman filters when there is a failure-free reserved measurement channel are proved.
Keywords-Dynamic system, Information theory, Kalman Filter, Multichannel measurement systems

1. INTRODUCTION

Application of algorithms for Kalman filtration in measurement science is very promising, since it permits efficient evaluation of not only measured parameters (the filtration problem), but also those parameters than cannot be measured directly (the identification problem). Here Kalman filtration made it possible to obtain optimal evaluations of state vectors and identifiable parameters of an object in real time in the presence of both internal and external noise.

The prospects of the use of the indicated mathematical apparatus for automated processing of measurement data for solution of metrological problems stems form the appearance of highly productive computational facilities (personal computers, measurement and calibration systems, etc.) We must also note the recently developed so called suboptimal Kalman filters, which, although yielding results that are somewhat inferior to those from optimal filters, can be implemented with less computational expense. All of this provides a basis for efficient use of Kalman filtration for processing measurement data for solution of metrological problems.

As a result of the current rapid development of microprocessor technology, it is now practical to use parallel measurement systems that make it possible to obtain information on the state vector of a system for several simultaneous measurements. Accordingly, algorithms have been developed for parallel evaluation of the parameters of dynamic systems that use a mathematical model of the motion of the system, and the results of measurements in several parallel measurement channels (parallel Kalman filtration). A number of such algorithms are presented in the literature [1-6]. In spite of the great variety of the parallel Kalman filter algorithms, till to the present, questions of informativness of parallel Kalman filters are not investigated.

In these algorithms concurrent processing of all available information made it possible to increase the accuracy in evolution of state vectors and the reliability of the processing process, even though this is accompanied by a substantial increase in the amount of computing required. This last factor immediately raises the problem of possibly reserving measurement systems in order to accomplish parallel estimation, so it is important to investigate the informativeness of parallel Kalman filters.

2. STATEMENT OF THE PROBLEM

Consider a dynamic system that is specified by an equation of state of the form

$$x(k+1)=\Phi(k+1,k)x(k)+w(k),$$

where $x(k)$ is (*nx1*)-dimensional state vector of the system, $\Phi(k+1,k)$ is an (*nxn*) transition matrix, $w(k)$ is an (*nx1*) –dimensional vector of Gaussian noise with zero mean and correlation matrix $E\left[w(k)w^T(j)\right]=Q(k)\delta(k)$; $E|\cdot|$ indicates the mathematical expectation, indicates transposition, and is the Kronecker symbol.

The output coordinates of the system are observed with a multichannel system consisting of M channels, where the equation of observations for the i-th channel is of the form

$$y_i(k)=H_i(k)x(k)+\upsilon_i(k),$$

where $y_i(k)$ is an (*sx1*)-dimensional vector of measurements of the *i*-th measurement channel, $H_i(k)$ is an $(s\times n)$ matrix of measurements for the *i*-th channel, $\upsilon_i(k)$ is an (*sx1*)-dimensional vector of Gaussian random noise in the measurements with zero mean and correlation matrix $E\left[v(k)v^T(j)\right]=R(k)\delta(k)$; and there are no correlations between the different measurement channels.

The state vector of the system can be estimated by using a parallel multichannel Kalman filter [1], which is characterized by:

$$\hat{x}(k/k)=\hat{x}(k/k-1)+\sum_{i=1}^{M}P(k/k)H_i^T(k)R_{ii}^{-1}(k)\left[y_i(k)-H_i(k)\hat{x}(k/k-1)\right], \quad (1)$$

where $\hat{x}(k/k-1)$ is an estimate for an extrapolation (prediction)one step ahead;

The correlation matrix for the filtration error is given as,

$$P^{-1}(k/k)=P^{-1}(k/k-1)+\sum_{i=1}^{M}H_i^T(k)R_{ii}^{-1}(k)H_i(k) \quad . \quad (2)$$

The correlation matrix for the extrapolation error is calculated as,

$$P(k/k-1)=\Phi(k,k-1)P(k-1/k-1)\Phi^T(k,k-1)+Q(k-1), \quad (3)$$

The main purpose is to investigate the informativeness of the parallel Kalman filter (1)-(3).

3. SOLUTION OF THE PROBLEM

It is possible to see from (1)-(3) that an optimal estimate for the case under consideration is obtained when simultaneous weighted summation of the innovation processes $\Delta i(k/k-1)=y_i(k)-H_i(k)\hat{x}(k/k-1)$ is used in all channels with a common extrapolator. As (1) implies, the weighting coefficients in this case are inversely proportional to the dispersions of the noise in the measurements of each channel. If an (*M+1*)-th failure free channel is added, then one can determine how the measurements of a given channel affect the informativeness of the filter.

It is well known, in the information theory that the quantity of information obtained from a measurement can be expressed in the following form as,

$$I = H(x) - H(x/z), \qquad (4)$$

where H(x) is the initial or *a priori* entropy of the measured function, and *H(x/z)* is the conditional (*a posteriori*) entropy, i.e., the entropy of the measured function after obtaining the measurement z.

In general, the quantity of information about a measured function that is made available by a measurement is equal to the reduction of the indeterminacy in the measured function, i.e., equal to the difference of the unconditional and conditional entropies.

Equation (4) shows how information grows as the result of measurement. If the conditional entropy exceeds the *apriori* entropy then the growth in the information can also be negative. Measurements are informative only if $I > 0$.

The a posteriory entropy *H(x/z)* is defined as the measurement error. In the case under consideration, *H(x/z)* depends on the method used to process measurement results. An optimal Kalman filter provides an estimate with a mathematical expectation equal to the estimated value, and a Gaussian distribution.

The quantities of information obtained by estimation with M – channel and (M+1) –channel Kalman filters are given, respectively, by the formulas

$$I\left[x(k), Y_k^M\right] = H\left[x(k)\right] - H\left[x(k)\,|Y_k^M\right]$$
$$I\left[x(k), Y_k^{M-1}\right] = H\left[x(k)\right] - H\left[x(k)\,|Y_k^{M+1}\right]$$

where

$$Y_k^M = \left[y_1(1), y_1(2), ..., y_1(k), y_2(1), y_2(2), ..., y_2(k), ..., y_M(1), y_M(2), ..., y_M(k)\right];$$
$$Y_k^{M-1} = \begin{bmatrix} y_1(1), y_1(2), ..., y_1(k), y_2(1), y_2(2), ..., y_2(k), ..., y_M(1), y_M(2), ..., \\ y_M(k), y_{M-1}(1), y_{M-1}(2), ..., y_{M-1}(k) \end{bmatrix}; \qquad (5)$$

$H\left[x(k)|Y_k^M\right]$ - is the entropy of the estimated state vector under the condition that it is realised in the form of measurements Y_k^M and $H\left[x(k)|Y_k^{M+1}\right]$ is the entropy of the estimated state vector under the condition that it is realised in the form of measurements Y_k^{M+1}.

Introduction of the functional

$$\Delta I^{M+1}(k) = I\left[x(k), Y_k^{M+1}\right] - I\left[x(k), Y_k^M\right] = H\left[x(k)|Y_k^M\right] - H\left[x(k)|Y_k^{M+1}\right] \qquad (6)$$

which is a quality criterion for the parallel multichannel system under consideration, characterises the informativeness of the (M+1)–th reserve channel. The quantity of information obtained by estimation is directly related to the accuracy of the estimate.

Theorem 1. In estimating the parameters of a dynamic system with a Kalman filter, the change in information obtained by addition of a reserved measurement channel is

$$\Delta I^{M+1}(k) = \frac{1}{2}\left\{\ln\left[\det P_M(k/k)\right] - \ln\left[\det P_{M+1}(k/k)\right]\right\} \qquad (7)$$

where $P_M(k/k)$ and $P_{M+1}(k/k)$ are the covariance matrices of the error in the estimate without and with an (M+1) –th reserved measurement channel.

Proof. Indeed, substituting the values of the entropies $H\left[x(k)|Y_k^M\right]$ and $H\left[x(k)|Y_k^{M+1}\right]$ into Eq.(6) and taking into account that the probability densities $p\left[x(k)|Y_k^M\right]$ and $p\left[x(k)|Y_k^{M+1}\right]$ that appear in them are Gaussian and constitute recursively computed estimates from Kalman filtration of the state vector and the correlation matrices of the error estimates [7] as,

$$p\left[x(k)|Y_k^M\right] = N\left\{\hat{x}_M(k/k), P_M(k/k)\right\};$$

and

$$p\left[x(k)|Y_k^{M+1}\right] = N\left\{\hat{x}_{M+1}(k/k), P_{M+1}(k/k)\right\}$$

indicates the validity of Eq. (7).

Theorem 2. When Kalman filtration (1)-(3) is used, the change in information obtained by addition of a reserved perfect information channel is positive:

$$\Delta l^{M+1}(k) > 0.$$

Proof. In a parallel multichannel Kalman filter, the matrices $P^{-1}(k/k)$ and $P_{M+1}^{-1}(k/k)$ can be written explicitly as,

$$P_M^{-1}(k/k) = p^{-1}(k/k-1) + \sum_{i=1}^{M} H_i^T(k) R_{ii}^{-1}(k) H_i(k); \tag{8}$$

and

$$P_{M+1}^{-1}(k/k) = p^{-1}(k/k-1) + \sum_{i=1}^{M+1} H_i^T(k) R_{ii}^{-1}(k) H_i(k) =$$
$$= p^{-1}(k/k-1) + \sum_{i=1}^{M} H_i^T(k) R_{ii}^{-1}(k) H_i(k) + H_{M+1}^T(k) \times R_{(M+1)(M+1)}^{-1}(k) H_{M-1}(k) \tag{9}$$

respectively.

Since the matrix R_{ii} is nondegenerate and positive definite, R_{ii}^{-1} is also positive definite [8]. It follows that the matrix $H_{M+1}^T(k) \times R_{(M+1)(M+1)}^{-1}(k) H_{M-1}(k)$ is the positive definite. Finally, subs traction of Eq. (8) from Eq. (9) leads to

$$P_{M+1}^{-1}(k/k) - P_M^{-1}(k/k) = H_{M+1}^T(k) \times R_{(M+1)(M+1)}^{-1}(k) H_{M-1}(k) \geq 0 \tag{10}$$

It is possible to write $A \succ B$ if the matrix A – B is positive semidefinite. Here, one can now write Eq.(10) in the more compact form as,

$$P_{M+1}^{-1}(k/k) \succ P_M^{-1}(k/k). \tag{11}$$

On the other hand, it is possible to present the following theorem from matrix theory without proof [8].

Let the matrices A and B be positive definite with

a) A $\succ$ B if and only if $B^{-1} \succ A^{-}$;
b) If A $\succ$ B, then det A $\geq$ det B.

In view of Eq. (11) and (a) of the theorem above, the following expression can be written

$$P_M(k/k) P \succ_{M+1} (k/k), \quad (12)$$

which asserts that the matrix $P_M(k/k)$ - $P_{M+1}(k/k)$ is positive semidefinite.

In view of Eq. (12) and (b) of the theorem above one can write that

$$\det P_M(k/k) \geq \det P_{M+1}(k/k) \quad (13)$$

Now the desired inequality $\Delta I^{M+1}(k) > 0$ follows from Eqs. (13) and (7).

4. CONCLUSION

The above discussion shows that the informativeness of a Kalman filter with a perfect reserved measurement channel is greater than in the case without such a channel. Here the increase in the information content resulting from addition of the (M+1)th channel is positive and determined by Eq. (6).

The current use of microprocessor computer technology in metrological practice provides the necessary prerequisites for effective application of reserved measurement channels in problems on estimating parameters of dynamic systems. The cost of the information increase in such systems lies primarily in the area of programming and, in practice, presents no additional hardware requirements.

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