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ON THE PROPERTIES OF (k+1)-DIMENSIONAL TIME-LIKE RULED SURFACES WITH THE SPACE-LIKE GENERATING SPACE IN THE MINKOWSKI SPACE IR_1^n

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Abstract-The purpose of this paper is to introduce a summary of known results and the definition of the time-like ruled surface with the space-like generating space in the Minkowski space IR_1^n, and to present some characteristic results related with minimality and total developability of the ruled surface in the n-dimensional Minkowski space IR_1^n .

Keywords-Time-Like Surface, Minkowski Space

1. INTRODUCTION

We will assume throughout this paper that all manifolds, maps, vector field, etc. ... are differentiable of class C^∞.

First of all, we give some properties of a general submanifold M of the Minkowski n-space IR_1^n, [1]. Let $\overline{D}$ be a Levi-Civita connection of IR_1^n and D be a Levi-Civita connection of M. If $X,Y \in \chi(M)$ and V is the second fundamental tensor of M, we have by decomposing $\overline{D}_X Y$ in tangential and normal components:

$$\overline{D}_X Y = D_X Y + V(X,Y). \qquad (1.1)$$

The equation (1.1) is called Gauss equation.

If ζ is any normal vector field on M, we find the Weingarten equation by decomposing $\overline{D}_X \zeta$ into tangential component and normal components as

$$\overline{D}_X \zeta = -A_\zeta(X) + D_X^\perp \zeta . \qquad (1.2)$$

A_ζ determines a self-adjoint linear map at each point and $D^\perp$ is a metric connection in the normal bundle $\chi^\perp(M)$. In this paper, we note that A_ζ will be used for the linear map and the corresponding matrix of the linear map.

If the metric tensor of IR_1^n is denoted by $\langle\,,\rangle$, from the equation (1.1) and (1.2), it follows that

$$\langle V(X,Y),\zeta\rangle = \langle A_\zeta(X),Y\rangle \qquad (1.3)$$

If $\zeta_1,\zeta_2,\ldots,\zeta_{n-m}$ constitute an orthonormal basis of $\chi^\perp(M)$, then we set

$$V(X,Y) = \sum_{j=1}^{n-m}\langle A_\zeta(X),Y\rangle\zeta_j . \qquad (1.4)$$

The mean curvature H of M at the point P is given by

$$H = \sum_{j=1}^{n-m} \frac{trA_{\zeta_j}}{\dim M} \zeta_j \,. \tag{1.5}$$

For every $X_i \in \chi(M)$, $1 \le i \le 4$, the 4th order covariant tensor field defined by R as

$$R(X_1, X_2, X_3, X_4) = \langle X_1, R(X_3, X_4) X_2 \rangle$$

is called the Riemann curvature tensor field and its value at a point $P \in M$, is called Riemann curvature of M at the point P.

If V is the second fundamental tensor, then we have

$$\langle Y, R(X,Y)X \rangle = \langle V(X,X), V(Y,Y) \rangle - \langle V(X,Y), V(X,Y) \rangle . \tag{1.6}$$

Let $\prod$ be a tangent plane of M at P. For all $X_P, Y_P \in \prod$, the real function K defined by

$$K(X_P, Y_P) = \frac{\langle R(X_P, Y_P) X_P, Y_P \rangle}{\langle X_P, X_P \rangle \langle Y_P, Y_P \rangle - \langle X_P, Y_P \rangle^2} \tag{1.7}$$

is called the section curvature function. $K(X_P, Y_P)$ is called the sectional curvature of M at P.

Let R be the Riemann curvature tensor and $\{e_1, e_2, \dots, e_m\}$ be a system of orthonormal basis of $T_M(P)$. The tensor field S defined in the form

$$S(X,Y) = \sum_{i=1}^{m} \varepsilon_i \langle R(X, e_i) Y, e_i \rangle \tag{1.8}$$

is called the Ricci curvature tensor field and the value of $S(X,Y)$ at $P \in M$ is also called the Ricci curvature, where

$$\varepsilon_i = \langle e_i, e_i \rangle \ , \ \varepsilon_i = \begin{cases} -1 & , \ if \ \ e_i \ \ time-like, \\ +1 & , \ if \ \ e_i \ \ space-like. \end{cases}$$

The real number r_{sk} defined in the form

$$r_{sk} = \sum_{i \ne j} K(e_i, e_j) = 2 \sum_{i<j} K(e_i, e_j) \tag{1.9}$$

is called the scalar curvature tensor field of M.

Let V be the second fundamental tensor of M. If

$$V(X,X) = 0 \tag{1.10}$$

for $X \in \chi(M)$, then X is called asymptotic vector field on M. If

$$V(X,Y) = 0 \tag{1.11}$$

for all $X, Y \in \chi(M)$, then M is totally geodesic.

Let M be a (k+1)-dimensional ruled surface in IR_1^n. Then M can be locally represented by

$$\phi(s, u_1, u_2, \dots u_k) = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s), \quad u_i \in IR, \quad 1 \le i \le k. \tag{1.12}$$

If the generating space $E_k(s) = sp\{e_1, e_2, \dots, e_k\}$ of M is *a space-like subspace* and *the base curve α is time-like*, then this surface is called the (k+1)-dimensional time-like ruled surface in IR_1^n,[2].

If

$$rank[e_0, e_1, \ldots, e_k, \overline{D}_{e_0} e_1, \ldots \overline{D}_{e_0} e_k] = 2k - m \tag{1.13}$$

at each point P of M, then M is called $\mathrm{m-developable}$. If $m = -1$, then generalized the time-like ruled surface M is called as non-developable. If $m = k-1$, M is called as total developable, where e_0 is the tangent vector of the base curve.

Suppose that $\{e_0, e_1, \ldots, e_k\}$ is an orthonormal base field of the tangential bundle $\chi(M)$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_{n-k-1}\}$ an orthonormal base field of the normal bundle $\chi^{\perp}(M)$. Then an orthonormal base field of $\chi(IR_1^n)$ is

$$\{e_0, e_1, \ldots, e_k, \zeta_1, \ldots, \zeta_{n-k-1}\}.$$

If we write the Weingarten derivative equation for the base vectors ζ_j we have

$$\overline{D}_{e_i} \zeta_j = -A_{\zeta_j}(e_i) + D^{\perp}_{e_i} \zeta_j \tag{1.14}$$

or

$$\begin{aligned} \overline{D}_{e_0} \zeta_j &= a_{00}^j e_0 + \sum_{r=1}^{k} a_{0r}^j e_r + \sum_{s=1}^{n-k-1} b_{0s}^j \zeta_s, \quad 1 \le j \le n-k-1 \\ \overline{D}_{e_i} \zeta_j &= a_{i0}^j e_0 + \sum_{r=1}^{k} a_{ir}^j e_r + \sum_{s=1}^{n-k-1} b_{is}^j \zeta_s, \quad 1 \le i \le k . \end{aligned} \tag{1.15}$$

From the above derivative equation we have

$$A_{\zeta_j} = -\begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0k}^j \\ -a_{01}^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_{0k}^j & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)} . \tag{1.16}$$

The Riemann curvature of the 2-dimensional cross section spanned by the vectors $(e_i)\big|_P$, $1 \le i \le k$, of M and $(e_0)\big|_P$ can be given by

$$K(e_i, e_0) = \left\langle \overline{D}_{e_i} e_0, \overline{D}_{e_i} e_0 \right\rangle = \sum_{j=1}^{n-k-1} \left(a_{0i}^j\right)^2 . \tag{1.17}$$

The mean curvature of M is

$$H = -\frac{1}{k+1} V(e_0, e_0) . \tag{1.18}$$

2. ON THE PROPERTIES AND SOME CHARACTERIZATION OF (k+1)-DIMENSIONAL TIME-LIKE RULED SURFACES WITH THE SPACE-LIKE GENERATING SPACE IN THE MINKOWSKI SPACE.

Theorem 1 Let M be $(k+1)$-dimensional time-like ruled surface and $\{e_1, e_2, \ldots, e_k\}$ be an orthonormal base field of the space-like generating space $E_k(s)$. Then the lines corresponding to $e_1, e_2, \ldots, e_k$ are asymptotics and geodesics of M .

Proof : Since the lines corresponding to the orthonormal base field vectors $e_1, e_2, \ldots, e_k$ of the space-like generating space $E_k(s)$ are geodesics of IR_1^n, we have

$$\overline{D}_{e_i} e_i = 0 \quad , 1 \le i \le k .$$

From (1.1) we have

$$D_{e_i} e_i = -V(e_i, e_i)$$

and thus

$$D_{e_i} e_i = 0, \qquad V(e_i, e_i) = 0 .$$

Therefore the lines corresponding to $e_1, e_2, \ldots, e_k$ are asymptotics and geodesics of M .

Theorem 2 M is total developable iff $\overline{D}_{e_i} e_0 = 0$, $1 \le i \le k$.

Proof : Let $\{e_0, e_1, \ldots, e_k\}$ be an orthonormal basis of M and M be total developable. Since the system $\{e_0, e_1, \ldots, e_k\}$ is linearly independent, $\overline{D}_{e_i} e_0$ has no component in the normal bundle $\chi^{\perp}(M)$, that is $V(e_i, e_0) = 0$.

We know that

$$\overline{D}_{e_0} e_i = V(e_0, e_i). \tag{2.1}$$

Since V is symmetric, from (2.1) we have

$$\overline{D}_{e_i} e_0 = 0, \quad 1 \le i \le k .$$

Conversely, assume that $\overline{D}_{e_i} e_0 = 0$. By (1.1) and (2.1) we have $V(e_i, e_0) = 0$. If we set this in the Gauss equation, we find

$$\overline{D}_{e_0} e_i = D_{e_0} e_i .$$

and

$$\overline{D}_{e_0} e_i \in sp\{e_0, e_1, \ldots, e_k\}$$

Thus we observe that

$$rank[e_0, e_1, \ldots, e_k, \overline{D}_{e_0} e_1, \overline{D}_{e_0} e_2, \ldots, \overline{D}_{e_0} e_k] = k + 1 .$$

Theorem 3 M is total developable and minimal iff M is totally geodesic.

Proof: We assume that M is total developable and minimal. If $X, Y \in \chi(M)$, we have

$$X = \sum_{i=1}^{k} a_i e_i + a e_0, \qquad Y = \sum_{j=1}^{k} b_j e_j + b e_0$$

Therefore we find

$$V(X,Y) = \sum_{i=1}^{k} (a_i b + b_i a) V(e_0, e_i) + ab V(e_0, e_0) + \sum_{i,j=1}^{k} a_i b_j V(e_i, e_j).$$

Since $V(e_i, e_j) = 0$ and M is minimal and total developable we have

$$V(X,Y) = 0 , \qquad \text{for all } X, Y \in \chi(M).$$

Conversely, let $V(X,Y) = 0$, for all $X, Y \in \chi(M)$. Then we have the following relations:

$$V(e_0, e_i) = 0 , \ V(e_0, e_0) = 0 \ \text{ and } \ V(e_i, e_j) = 0 , \qquad 1 \le i, j \le k$$

By using these equations and (2.1), we find $\overline{D}_{e_i} e_0 = 0$ and so, M is total developable. Moreover, $V(e_0, e_0) = 0$ implies that $H = 0$. Therefore M is minimal.

Let $\{e_0, e_1, \ldots, e_k\}$ an orthonormal basis of $\chi(M)$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_{n-k-1}\}$ an orthonormal basis of $\chi^{\perp}(M)$. Moreover, we can give covariant derivative equations of the orthonormal basis $\{e_0, e_1, \ldots, e_k, \zeta_1, \ldots, \zeta_{n-k-1}\}$ of $\chi(IR_1^n)$ as follows:

$$\overline{D}_{e_0} e_r = \sum_{i=0}^{k} c_{ri} e_i + \sum_{m=1}^{n-k-1} c_{r(k+m)} \zeta_m, \qquad 0 \le r \le k$$
$$\overline{D}_{e_0} \zeta_j = \sum_{i=0}^{k} c_{(k+j)i} e_i + \sum_{m=1}^{n-k-1} c_{(k+j)(k+m)} \zeta_m, \qquad 1 \le j \le n-k-1. \tag{2.2}$$

If we calculate the coefficient c_{st}, $0 \le s, t \le n-1$, and write the equation (2.3) in the matrix form we obtain:

$$\begin{bmatrix} \overline{D}_{e_0} e_0 \\ \overline{D}_{e_0} e_1 \\ \vdots \\ \overline{D}_{e_0} e_k \\ \overline{D}_{e_0} \zeta_1 \\ \vdots \\ \overline{D}_{e_0} \zeta_{n-k-1} \end{bmatrix} = \begin{bmatrix} 0 & c_{01} & \cdots & c_{0k} & c_{0(k+1)} & \cdots & c_{0(n-1)} \\ c_{01} & 0 & \cdots & c_{1k} & c_{1(k+1)} & \cdots & c_{1(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{0k} & -c_{1k} & \cdots & 0 & c_{k(k+1)} & \cdots & c_{k(n-1)} \\ c_{0(k+1)} & -c_{1(k+1)} & \cdots & -c_{k(k+1)} & 0 & \cdots & c_{(k+1)(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{0(n-1)} & -c_{1(n-1)} & \cdots & -c_{k(n-1)} & -c_{(k+1)(n-1)} & \cdots & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_k \\ \zeta_1 \\ \vdots \\ \zeta_{n-k-1} \end{bmatrix}. \tag{2.3}$$

By using the equation (2.3) we can give the following theorem.

Theorem 4 Let M be a $(k+1)-$dimensional time-like ruled surface in IR_1^n, $\{e_1, e_2, \ldots, e_k\}$ be an orthonormal base field of the space-like generating space $E_k(s)$ and let the base curve $\alpha(s)$ be an orthogonal trajectory of $E_k(s)$. Then the following propositions are equivalent:

(i) M is total developable,

(ii) The Riemanian curvature $K(e_i, e_0)$ of M is zero, $1 \le i \le k$,

(iii) In the equation (2.3) $c_{rs} = 0, \quad 1 \le i \le k, \quad k+1 \le s \le n-1,$

(iv) $A_{\zeta_j}(e_i) = 0, \quad 1 \le i \le k, \quad 1 \le j \le n-k-1$,

(v) $\overline{D}_{e_0} e_i \in \chi(M)$.

Proof:

$(i \Rightarrow ii)$: We assume that M is total developable. Then by the Theorem 2 and the equation (1.17) we find

$$K(e_i, e_0) = 0.$$

$(ii \Rightarrow iii)$: Let $K(e_i, e_0) = 0.$

From (1.15) and (1.16) we find

$$\left\langle \overline{D}_{e_0}\zeta_j, e_i \right\rangle = 0.$$

This equation shows that $\overline{D}_{e_0}\zeta_j$ has no component in the directions of $e_1, e_2, \cdots, e_k$. Hence we have

$$c_{rs} = 0$$

in the equation (2.3).

$(iii \Rightarrow iv)$: Let's assume that $c_{rs} = 0$. By (2.3) we obtain

$$\left\langle \overline{D}_{e_0}\zeta_j, e_i \right\rangle = -\varepsilon_i c_{is} = 0, \qquad 1 \le j \le n-k-1.$$

Thus, from (1.17), it is seen that

$$\left\langle \overline{D}_{e_0}\zeta_j, e_i \right\rangle = \varepsilon_i a^j_{0i}$$

and

$$a^j_{0i} = 0.$$

By (1.17) we know that

$$\left\langle \overline{D}_{e_i}\zeta_j, e_r \right\rangle = 0 .$$

Then from last two equations, we obtain

$$A_{\zeta_j}\left(e_i\right) = 0.$$

$(iv \Rightarrow v)$: Let $A_{\zeta_j}\left(e_i\right) = 0.$

By (1.17) we have

$$a^j_{0i} = 0 .$$

and $\overline{D}_{e_0}\zeta_j$ has no component in the directions of $e_1, e_2, \cdots, e_k$, i.e.

$$c_{rs} = 0.$$

Then from (2.3) we have

$$\left\langle \overline{D}_{e_0}\zeta_j, e_i \right\rangle = 0$$

Since

$$\left\langle \overline{D}_{e_0}\zeta_j, e_i \right\rangle = -\left\langle \overline{D}_{e_0}e_i, \zeta_j \right\rangle = 0$$

we may write

$$\overline{D}_{e_0}e_i \in \chi(M).$$

$(v \Rightarrow i)$: Let $\overline{D}_{e_0}e_i \in \chi(M)$. Thus we have

$$\overline{D}_{e_0}e_i \in sp\{e_0, e_1, \ldots, e_k\}$$

or

$$rank[e_0, e_1, \ldots, e_k, \overline{D}_{e_0}e_1, \ldots \overline{D}_{e_0}e_k] = k+1 .$$

This means that M is total developable.

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