

HOPF POINT ANALYSIS FOR RATIO-DEPENDENT FOOD CHAIN MODELS

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Abstract - In this paper periodic and quasi-periodic behavior of a food chain model with three trophic levels are studied. Michaelis-Menten type ratio-dependent functional response is considered. There are two equilibrium points of the system. It is found out that at most one of these equilibrium points is stable at a time. In the parameter space, there are passages from instability to stability, which are called Hopf bifurcation points. For the first equilibrium point, it is possible to find bifurcation points analytically and to prove that the system has periodic solutions around these points. However for the second equilibrium point the computation is more tedious and bifurcation points can only be found by numerical experiments. It has been found that around these points there are periodic solutions and when this point is unstable, the solution is an enlarging spiral from inside and approaches to a limit cycle.

Keywords: Food chain models, Hopf bifurcation, limit cycles, periodic solutions.

1. INTRODUCTION

The term “ratio-dependent predation” is introduced in [1] to describe situations in which the feeding rates of predators depend on the ratio of the number of preys to the number of predators rather than on prey density alone, as is the case in most classical models. One advantage of the ratio dependence is that they prevent paradoxes of enrichment and biological control predicted by classical models [2,3].

Experimental observations [4] suggest that prey dependent models are appropriate in homogeneous situations and ratio-dependent models are good in heterogeneous cases. By many investigators [4, 5] it has also been concluded that

natural systems are closer to the models with ratio dependence than to the ones with prey density dependence [6].

Generally, a ratio-dependent predator-prey model leads to a system of nonlinear ordinary differential equations. The classical food chain models with only two trophic levels are shown to be insufficient to produce realistic dynamics [12-16]. Therefore we consider the following three trophic levels food chain model with ratio-dependence which is a simple relation between the populations of the three species: z prey on y and only y , and y prey on x and nutrient recycling is not accounted for. After non dimensionalization we have the following system:

$$\begin{aligned} \dot{x} &= \left(1 - x - \frac{c_1 y}{x+y}\right) x, & x(0) > 0, \\ \dot{y} &= \left(\frac{m_1 x}{x+y} - d_1 - \frac{c_2 z}{y+z}\right) y, & y(0) > 0, \\ \dot{z} &= \left(-d_2 + \frac{m_2 y}{y+z}\right) z, & z(0) > 0, \\ c_1 &= m_1/(\eta_1 a_1 r), \quad c_2 = m_2/(\eta_2 a_2 r). \end{aligned} \tag{1}$$

Where x, y, z stand for the non dimensional population density of the prey, predator and top predator respectively. For $i=1,2$, η_i, m_i, a_i, d_i are the yield constants, maximal predator growth rates, half-saturation constants and predators' death rates, r is the prey intrinsic growth rate.

2. EQUILIBRIUM POINTS

Equilibrium points are the solutions of the nonlinear algebraic system of equations [17,18],

$$\begin{aligned} \left(1 - x - \frac{c_1 y}{x+y}\right) x &= 0, \\ \left(\frac{m_1 x}{x+y} - d_1 - \frac{c_2 z}{y+z}\right) y &= 0, \\ \left(-d_2 + \frac{m_2 y}{y+z}\right) z &= 0, \quad x(0), y(0), z(0) > 0. \end{aligned} \tag{2}$$

Considering the nonnegative ness of the parameters and unknowns, we get two equilibrium points. One of them is of the form $E_0(\bar{x}, \bar{y}, 0)$ and the other is $E_1(x^*, y^*, z^*)$.

The Equilibrium Point $E_0(\bar{x}, \bar{y}, 0)$

The first equilibrium point $E_0(\bar{x}, \bar{y}, 0)$ with

$$\bar{x} = \frac{m_1(1-c_1)+c_1 d_1}{m_1}, \quad \bar{y} = \frac{m_1-d_1}{d_1} \bar{x}, \quad \bar{z} = 0 \tag{3}$$

is a nonnegative equilibrium point of the system (1) if

$$m_1(1 - c_1) + c_1 d_1 > 0 \quad \text{and} \quad m_1 > d_1. \quad (4)$$

The Equilibrium Point $E_1(x^*, y^*, z^*)$

The second equilibrium point $E_1(x^*, y^*, z^*)$ with

$$x^* = 1 + c_1 B, \quad y^* = -(B/(1+B))x^*, \quad z^* = (m_2/d_2 - 1)y^*, \quad B = (c_2(m_2 - d_2) + m_2(d_1 - m_1))/(m_1 m_2) \quad (5)$$

is an interior equilibrium of the system (1) if

$$\max(-1, -1/c_1) < B < 0, \quad \text{and} \quad d_2 < m_2. \quad (6)$$

3. STABILITY OF EQUILIBRIUMS

The dynamical behavior of equilibrium points is studied by computation of the eigenvalues of the variational matrix J ;

$$\mathbf{J}(\mathbf{v}, \boldsymbol{\mu}) = \begin{bmatrix} 1 - 2x - \frac{c_1 y^2}{(x+y)^2} & -\frac{c_1 x^2}{(x+y)^2} & 0 \\ \frac{m_1 y^2}{(x+y)^2} & -d_1 + \frac{m_1 x^2}{(x+y)^2} - \frac{c_2 z^2}{(y+z)^2} & -\frac{c_2 y^2}{(y+z)^2} \\ 0 & \frac{m_2 z^2}{(y+z)^2} & -d_2 + \frac{m_2 y^2}{(y+z)^2} \end{bmatrix} \quad (7)$$

Where $\mathbf{v} = (x, y, z)$, $\boldsymbol{\mu} = (c_1, d_1, m_1, c_2, d_2, m_2)$, at each equilibrium point.

For the equilibrium point E_0 : The eigenvalues are:

$$\lambda_1 = \frac{1}{2m_1^2} (L + \sqrt{M}), \quad \lambda_2 = \frac{1}{2m_1^2} (L - \sqrt{M}), \quad \lambda_3 = -d_2 + m_2. \quad (8)$$

Where

$$\begin{aligned} L &= c_1(m_1^2 - d_1^2) - d_1 m_1(m_1 - d_1) - m_1^2, \\ M &= L^2 + 4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1). \end{aligned} \quad (9)$$

E_0 is a nonnegative equilibrium point of the system (1), if $m_1(1 - c_1) + c_1 d_1 > 0$ and $m_1 > d_1$. Hence one has $4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1) < 0$, and therefore

$$\lambda_1 \lambda_2 = \frac{L^2 - M}{4m_1^4} = -4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1) > 0. \quad (10)$$

That is the roots have the same sign if they are real. On the other hand if $L > 0$ one also has

$$\lambda_1 + \lambda_2 = \frac{L}{m_1^2} > 0. \quad (11)$$

Hence is $\lambda_1, \lambda_2 > 0$ if the roots are real and $\Re(\lambda_1) = \Re(\lambda_2) > 0$ if the two roots are complex conjugate. If $m_2 > d_2$, then $\lambda_3 > 0$ and in this case E_0 is a repeller point. If $m_2 < d_2$, one has $\lambda_3 < 0$ then E_0 is saddle point, that is, E_0 is unstable in both cases [19,20].

On the other hand, if $L < 0$, then $\lambda_1, \lambda_2 < 0$ if the two roots are real. $\Re(\lambda_1) = \Re(\lambda_2) < 0$ if the two roots are complex conjugate. If $m_2 > d_2$ then $\lambda_3 > 0$ and E_0 is saddle point. If $m_2 < d_2$, then $\lambda_3 < 0$ and E_0 is spiral node. In the latter case, the second equilibrium point $E_1(x^*, y^*, z^*)$ does not lie in the physical space. Hence the system can not have two stable equilibrium points for the same set of parameters.

For the equilibrium point E_1 :

It can be shown that the real parts of the roots of the cubic algebraic equation $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$ with real coefficients are all with negative real parts if and only if $A_1, A_2, A_3 > 0$ and $A_1A_2 > A_3$.

For the Jacobi matrix

$$\mathbf{J}(E_1, \mathbf{v}) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad (12)$$

one has

$$\begin{aligned} A_1 &= -a_{11} - a_{22} - a_{33}, \\ A_2 &= a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33} - a_{12}a_{21} - a_{23}a_{32}, \\ A_3 &= a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33}. \end{aligned} \quad (13)$$

Using equilibrium conditions obtained from (2), we see that

$$\begin{aligned} a_{11} &= x \left(-1 + \frac{c_1 y}{(x+y)^2} \right), \quad a_{12} = \frac{-c_1 x^2}{(x+y)^2}, \\ a_{21} &= \frac{m_1 x^2}{(x+y)^2}, \quad a_{22} = y \left(-\frac{m_1 x}{(x+y)^2} + \frac{c_2 z}{(y+z)^2} \right), \quad a_{23} = \frac{-c_2 y^2}{(y+z)^2}, \\ a_{32} &= \frac{m_2 z^2}{(y+z)^2}, \quad a_{33} = \frac{m_2 y z}{(y+z)^2}. \end{aligned} \quad (14)$$

It can be shown that the coefficients A_1, A_2, A_3 satisfy the inequalities $A_1, A_2, A_3 > 0$ and $A_1A_2 > A_3$ if

$$\frac{c_1 y^*}{(x^* + y^*)^2} < 1, \quad \frac{c_2 z^*}{(y^* + z^*)^2} < \frac{m_1 x^*}{(x^* + y^*)^2} \quad (15)$$

Therefore the characteristic equation of the Jacobi matrix (12) has roots with all negative real parts, and hence E_1 is a stable equilibrium point under these conditions [21].

4. HOPF BIFURCATION POINTS

When interested in periodic or quasi periodic behavior of a dynamical system, Hopf points are the points which are first to be considered. If we write the autonomous system (1) in the form

$$\dot{\mathbf{v}} = \mathbf{F}(\mathbf{v}, \boldsymbol{\mu}), \quad (16)$$

where

$$\mathbf{v} = (x, y, z), \quad \boldsymbol{\mu} = (c_1, d_1, m_1, c_2, d_2, m_2). \quad (17)$$

We say that an ordered pair $(\mathbf{v}_0, \boldsymbol{\mu}_0)$ is a Hopf bifurcation point if

1. $\mathbf{F}(\mathbf{v}_0, \boldsymbol{\mu}_0) = \mathbf{0}$,
2. $\mathbf{J}(\mathbf{v}, \boldsymbol{\mu})$ has two complex conjugate eigenvalues around $(\mathbf{v}_0, \boldsymbol{\mu}_0)$,
 $\lambda_{1,2} = a(\mathbf{v}, \boldsymbol{\mu}) \pm I b(\mathbf{v}, \boldsymbol{\mu})$,
3. $a(\mathbf{v}_0, \boldsymbol{\mu}_0) = 0, \nabla a(\mathbf{v}_0, \boldsymbol{\mu}_0) \neq \mathbf{0}, b(\mathbf{v}_0, \boldsymbol{\mu}_0) \neq 0$,
4. The third eigenvalue $\lambda_3(\mathbf{v}_0, \boldsymbol{\mu}_0) \neq 0$.

For the system (1), at two points $E_0(\bar{x}, \bar{y}, 0)$ and $E_1(x^*, y^*, z^*)$, one has $\mathbf{F}(\mathbf{v}_0, \boldsymbol{\mu}) = \mathbf{0}$.

I. For the equilibrium point E_0 :

At $E_0(\bar{x}, \bar{y}, 0)$ one has two complex conjugate eigenvalues if $M < 0$ in (8). The real parts of the eigenvalues are zero if $L = 0$ in the same expression:

$$M = 4m_1^2 d_1 (d_1 - m_1) (m_1 (1 - c_1) + c_1 d_1) < 0 \quad (18)$$

$$L = c_1 (m_1^2 - d_1^2) - d_1 m_1 (m_1 - d_1) - m_1^2 = 0, \quad (19)$$

Inequalities in (4) that guaranties the existence of $E_0(\bar{x}, \bar{y}, 0)$ as a real world point, also assures (18). (19) is satisfied if c_1 is chosen as

$$c_{10} = \frac{d_1 m_1 (m_1 - d_1) + m_1^2}{m_1^2 - d_1^2}. \quad (20)$$

The last condition $\lambda_3(\mathbf{v}_0, \boldsymbol{\mu}_0) \neq 0$ is satisfied if $d_2 \neq m_2$.

L is an increasing function of c_1 , hence E_0 is stable for $c_1 < c_{10}$ and unstable for $c_1 > c_{10}$. Hence the point $(\mathbf{v}_0, \boldsymbol{\mu}_0)$ corresponding to $c_1 = c_{10}$ is a Hopf bifurcation point.

4.1 Numerical Experiments

1. Periodic Solutions Around E_0

For $d_1 = 1.0$, $m_1 = 1.1$, c_{10} calculated from (20) is $c_{10} = 6.28571$. For the set of parameters

$$\boldsymbol{\mu}_0 = \{6.2857142857143, 11., 1., 2.1., 1.1, 2\} \quad (21)$$

The coordinates of E_0 becomes $\mathbf{v}_0 = \{0.428571, 0.0428571, 0.\}$. Eigenvalues are obtained as $\lambda_1 = 0.197385 I$, $\lambda_2 = -0.197385 I$, $\lambda_3 = -1$.

For an initial point $\{0.436, 0.044, 0.00000001\}$, very near to E_0 , the solution is periodic with period 31.94 as seen in Fig.1.

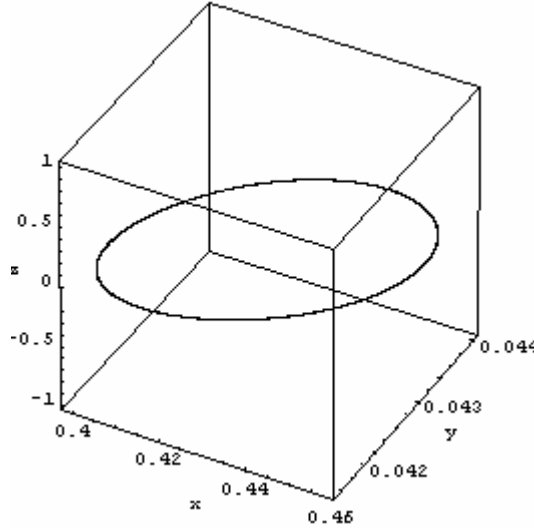


Fig.1. Solution for the parameter set $\boldsymbol{\mu}_0 = \{6.2857142857143, 11., 1., 2.1., 1.1, 2\}$ and initial data $\{0.436, 0.044, 0.00000001\}$.

2. E_0 is a spiral node

For the set of parameters

$$\boldsymbol{\mu}_0 = \{5., 11., 1., 2.1., 1.1, 2\} \quad (22)$$

The coordinates of E_0 becomes

$$\mathbf{v}_0 = \{0.545455, 0.0545455, 0.\} \quad (23)$$

Eigenvalues are obtained as $\lambda_1 = -0.11157 + 0.192714 I$, $\lambda_2 = -0.11157 - 0.192714 I$, $\lambda_3 = -1$. For an initial point $\{0.545, 0.054, 0.00001\}$, very near to E_0 , the solution is a spiral that terminates at the spiral node E_0 , as seen in Fig.2.

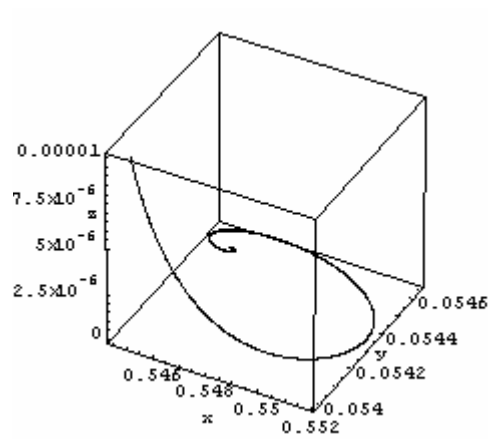


Fig.2. Solution for the parameter set $\mu_0 = \{5., 11., 1., 2.1., 1.1, 2\}$ and initial data $\{0.545, 0.054, 0.00001\}$ is a spiral terminating at E_0 .

3. At E_0 the solution is an enlarging spiral

For the set of parameters

$$\mu_0 = \{6.5, 11., 1., 2.1., 1.1, 2\} \quad (24)$$

The coordinates of E_0 becomes

$$v_0 = \{0.409091, 0.0409091, 0.\} \quad (25)$$

Eigenvalues are obtained as $\lambda_1 = 0.18595 + 0.191949 I$, $\lambda_2 = 0.18595 - 0.191949 I$, $\lambda_3 = -1$. For an initial point $\{0.405, 0.0405, 0.0000001\}$, very near to E_0 , the solution is a spiral that runs away of E_0 , as seen in Fig.3.

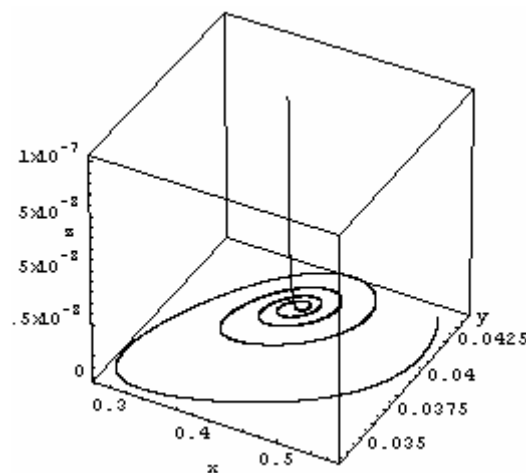


Fig.3. Solution for the parameter set $\mu_0 = \{6.5, 11., 1., 2.1., 1.1, 2\}$ and initial data $\{0.405, 0.0405, 0.0000001\}$ is a spiral running away of E_0 .

II. For the equilibrium point E_1 :

In a nine dimensional $(\mathbf{v}, \boldsymbol{\mu})$ space, even when we restrict ourselves to the point E_1 , the computational difficulties to find Hopf points are enormous. Hence we restrict ourselves to the hyper surface $(E_1, \boldsymbol{\mu}_m)$ with

$$\boldsymbol{\mu}_m = \{1., 11., 1., 1., 10., m_2\}. \quad (26)$$

It has been showed numerically that the only Hopf point is found when $m_2 = 2.0061353417583831$ in sixteen digits. E_1 is stable when $m_2 < 2.0061353417583831$ and unstable when $m_2 > 2.0061353417583831$.

1. Limit Cycle Around E_1

For $m_2 = 2.0061353417583831$,

$$\boldsymbol{\mu}_1 = \{1., 11., 1., 1., 10., 2.0061353417583831\} \quad (27)$$

the coordinates of E_1 and the corresponding eigenvalues are: $\mathbf{v}_1 = \{0.651682, 0.348318, 0.350455\}$, $\lambda_1 = 1.2895 I$, $\lambda_2 = -1.2895 I$, $\lambda_3 = -0.44617$. For an initial point $\{0.651682, 0.348318, 0.350455\}$, close to E_1 , the solution approaches to a limit cycle with period 4.87 as seen in Fig.4.

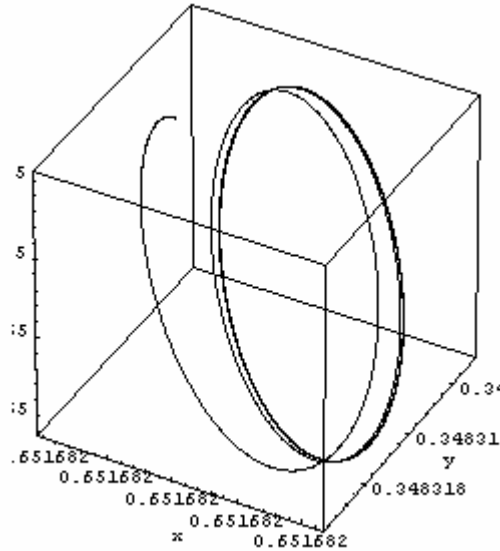


Fig.4. For initial data $\{0.651682, 0.348318, 0.350455\}$ and $\boldsymbol{\mu}_1 = \{1., 11., 1., 1., 10., 2.0061353417583831\}$. Solution has a limit cycle with period 4.87.

2. Spiral node at E_1

For $m_2 = 1.5 < 2.0061353417583831$, $\boldsymbol{\mu}_1 = \{1., 11., 1., 1., 10., 1.5\}$ the coordinates of E_1 and the corresponding eigenvalues are:

$$\mathbf{v}_1 = \{0.466667, 0.533333, 0.266667\}$$

$$\lambda_1 = -0.163679 + 1.19028 I, \lambda_2 = -0.163679 - 1.19028 I, \lambda_3 = -0.268198.$$

For an initial point $\{0.4, 0.5, 0.3\}$ the solution is a spiral terminating at E_1 , as in Fig.5.

3. Enlarging spiral at E_1

For $m_2 = 2.3 > 2.0061353417583831$, $\mu_1 = \{1., 11., 1., 1., 10., 2.3\}$ the coordinates of E_1 and the corresponding eigenvalues are:

$$v_1 = \{0.721739, 0.278261, 0.361739\};$$

$$\lambda_1 = 0.0695709 + 1.24092 I, \lambda_2 = 0.0695709 - 1.24092 I, \lambda_3 = -0.530371.$$

For an initial point close to E_1 , the solution is an enlarging spiral as seen in Fig.6.

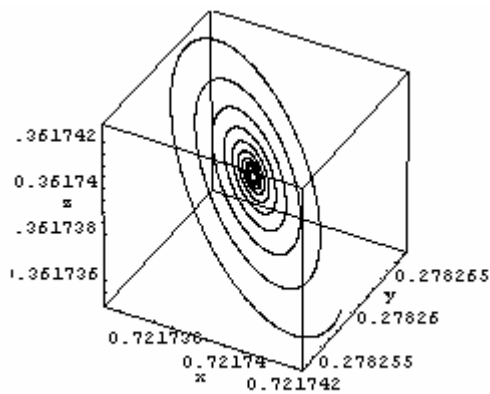


Fig.6. For initial data close to E_1 and parameters $\mu_1 = \{1., 11., 1., 1., 10., 2.3\}$, the solution is an enlarging spiral.

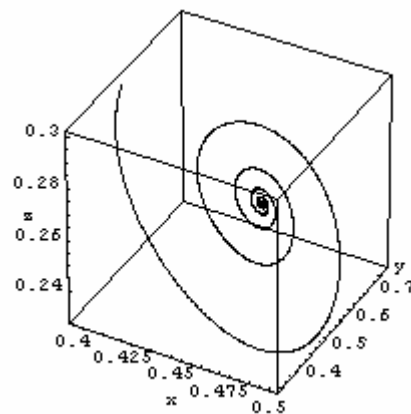


Fig.5. For initial data $\{0.4, 0.5, 0.3\}$ and $\mu_1 = \{1., 11., 1., 1., 10., 1.5\}$, the solution is spiral terminating at E_1 .

4. Limit Cycle around E_1

For $m_2 = 2.007 > 2.0061353417583831$, $\mu_1 = \{1., 11., 1., 1., 10., 2.007\}$ the coordinates of E_1 and the corresponding eigenvalues are $v_1 = \{0.651918, 0.348082, 0.350518\}$ and $\lambda_1 = 0.00022607 + 1.28942 I, \lambda_2 = 0.00022607 - 1.28942 I, \lambda_3 = -0.446435$. For an initial point at $\{0.7, 0.35, 0.5\}$, the solution enlarges by the time, and eventually reaches to a limit cycle with period 8.37 as shown in Fig. 7.

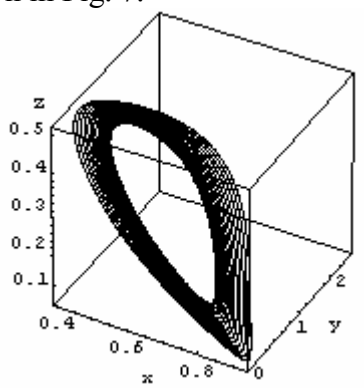


Fig.7. For initial data $\{0.7, 0.35, 0.5\}$ and parameter values $\mu_1 = \{1., 11., 1., 1., 10., 2.007\}$, the solution has a limit cycle with period 8.37.

5. VANISHING TOP PREDATOR

When the top predator vanishes, some nonlinear oscillatory phenomena for the first predator and pray occurs. Consider the system

$$\begin{aligned} \dot{x} &= \left(1 - x - \frac{c_1 y}{x+y}\right) x, \quad x(0) > 0, \\ \dot{y} &= \left(\frac{m_1 x}{x+y} - d_1\right) y, \quad y(0) > 0, \\ c_1 &= m_1 / (\eta_1 a_1 r). \end{aligned} \quad (28)$$

which is obtained deleting z from (1).

Considering the nonnegative ness of the parameters and unknowns, we get two equilibrium points. One of them is $C_0(1,0)$ and the other is $C_1(\bar{x}, \bar{y})$. The point $C_0(1,0)$ is always a positive equilibrium point.

The Equilibrium Point $C_1(\bar{x}, \bar{y})$

The second equilibrium point $C_1(\bar{x}, \bar{y})$ with

$$\bar{x} = \frac{m_1(1-c_1) + c_1 d_1}{m_1}, \quad \bar{y} = \frac{m_1 - d_1}{d_1} \bar{x}, \quad (29)$$

is a nonnegative equilibrium point of the system (28) if

$$m_1(1-c_1) + c_1 d_1 > 0 \quad \text{and} \quad m_1 > d_1. \quad (30)$$

Stability of Equilibriums

The dynamical behavior of equilibrium points is studied by computation of the eigenvalues of the variational matrix J_1 at each equilibrium point:

$$\mathbf{J}_1(\mathbf{v}, \boldsymbol{\mu}) = \begin{bmatrix} 1 - 2x - \frac{c_1 y^2}{(x+y)^2} & -\frac{c_1 x^2}{(x+y)^2} \\ \frac{m_1 y^2}{(x+y)^2} & -d_1 + \frac{m_1 x^2}{(x+y)^2} \end{bmatrix}, \quad \mathbf{v} = (x, y), \quad \boldsymbol{\mu} = (c_1, d_1, m_1), \quad (31)$$

For the equilibrium point $C_0(1,0)$:

The eigenvalues are $\lambda_1 = -1.0$, $\lambda_2 = 0.1$, and hence $C_0(1,0)$ is always a saddle point.

For the equilibrium point $C_1(\bar{x}, \bar{y})$: The eigenvalues are

$$\lambda_1 = \frac{1}{2m_1^2} (L + \sqrt{M}), \quad \lambda_2 = \frac{1}{2m_1^2} (L - \sqrt{M}) \quad (32)$$

Where

$$\begin{aligned} L &= c_1(m_1^2 - d_1^2) - d_1 m_1(m_1 - d_1) - m_1^2, \\ M &= L^2 + 4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1). \end{aligned} \quad (33)$$

$C_1(\bar{x}, \bar{y})$ is a nonnegative equilibrium point of the system (28) if $m_1(1 - c_1) + c_1 d_1 > 0$ and $m_1 > d_1$. Hence one has $4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1) < 0$, and therefore

$$\lambda_1 \lambda_2 = \frac{L^2 - M}{4m_1^4} = -4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1) > 0. \quad (34)$$

That is the roots have the same sign if they are real. On the other hand if $L > 0$ one also has

$$\lambda_1 + \lambda_2 = \frac{L}{m_1^2} > 0. \quad (35)$$

Hence, $\lambda_1, \lambda_2 > 0$ if the roots are real and $\Re(\lambda_1) = \Re(\lambda_2) > 0$ if the two roots are complex conjugate, and in this case $C_1(\bar{x}, \bar{y})$ is unstable [19]. If $L < 0$, then $\lambda_1, \lambda_2 < 0$ if the two roots are real. $\Re(\lambda_1) = \Re(\lambda_2) < 0$ if the two roots are complex conjugate, and $C_1(\bar{x}, \bar{y})$ is stable and is a spiral node. The state $L = 0$ corresponds to a Hopf bifurcation point.

Numerical Experiments

At $C_1(\bar{x}, \bar{y})$ one has two complex conjugate eigenvalues if $M < 0$ in (33). The real parts of eigenvalues are zero if $L = 0$ in the same expression:

$$M = 4m_1^2 d_1(d_1 - m_1)(m_1(1 - c_1) + c_1 d_1) < 0 \quad (36)$$

$$L = c_1(m_1^2 - d_1^2) - d_1 m_1(m_1 - d_1) - m_1^2 = 0, \quad (37)$$

Inequalities in (30) that guaranties the existence of $C_1(\bar{x}, \bar{y})$ as a real world point, also assures (36). (37) is satisfied if c_1 is chosen as

$$c_{10} = \frac{d_1 m_1(m_1 - d_1) + m_1^2}{m_1^2 - d_1^2}. \quad (38)$$

L is an increasing function of c_1 , hence $C_1(\bar{x}, \bar{y})$ is stable for $c_1 < c_{10}$ and unstable for $c_1 > c_{10}$. Hence points $(\mathbf{v}_0, \boldsymbol{\mu}_0)$ corresponding to $c_1 = c_{10}$ defines a hyper surface of Hopf bifurcation points in the five dimensional space $(\mathbf{v}, \boldsymbol{\mu})$.

Numerical Experiments

1. Periodic Solutions Around $C_1(\bar{x}, \bar{y})$

For $d_1 = 1.0$, $m_1 = 1.1$, c_{10} from (38) is $c_{10} = 6.2857142857142857142857142857$ in 28 digits. For the set of parameters

$$\boldsymbol{\mu}_0 = \{6.2857142857143, 1.0, 1.1\} \quad (39)$$

The coordinates of $C_1(\bar{x}, \bar{y})$ becomes $\mathbf{v}_0 = \{0.428571, 0.0428571\}$. Eigenvalues are obtained as $\lambda_1 = 0.197385 I$, $\lambda_2 = -0.197385 I$. Therefore $C_1(\bar{x}, \bar{y})$ is a center. For an initial point $\{0.429, 0.043\}$, near $C_1(\bar{x}, \bar{y})$, the solution is periodic with period 31.83 as seen in Fig.8.

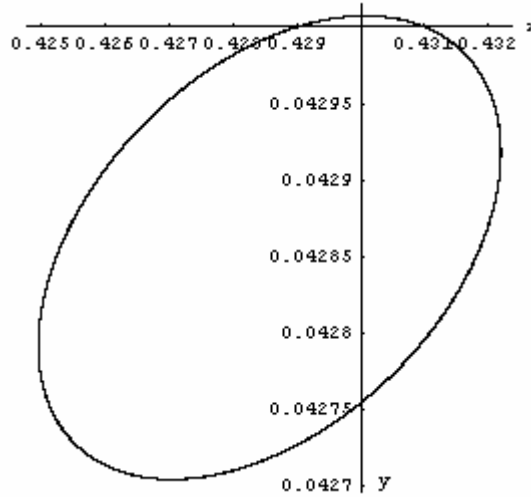


Fig.8. For initial data $\{0.429, 0.043\}$ and $\mu_0 = \{6.2857142857143, 1.0, 1.1\}$, the solution is periodic with period 31.83.

2. *Stable spiral point at $C_1(\bar{x}, \bar{y})$*

For the set of parameters

$$\mu_1 = \{5.0, 1.0, 1.1\}, \tag{40}$$

the coordinates of $C_1(\bar{x}, \bar{y})$ becomes $\mathbf{v}_1 = \{0.545, 0.0545\}$. Eigenvalues are obtained as $\lambda_1 = \{-0.11157 + 0.192714 I$, $\lambda_2 = -0.11157 - 0.192714 I\}$. Hence $C_1(\bar{x}, \bar{y})$ is a stable spiral point. For an initial point $\{0.545, 0.0545\}$ very close to $C_1(\bar{x}, \bar{y})$, the solution is a spiral as seen in Fig.9.

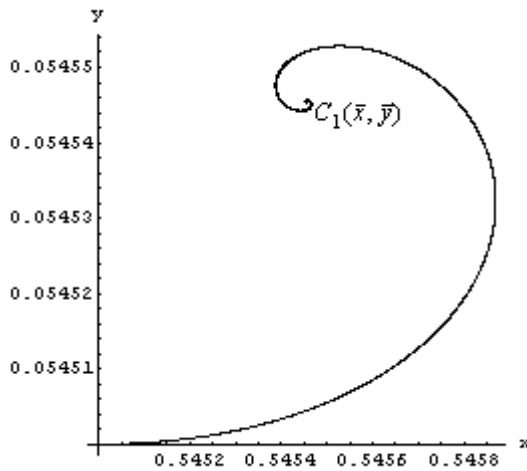


Fig.9. For initial data $\mathbf{v}_1 = \{0.545, 0.0545\}$ and $\mu_1 = \{5.0, 1.0, 1.1\}$, the solution is a spiral.

3. An unstable spiral point at $C_1(\bar{x}, \bar{y})$

For the set of parameters

$$\mu_2 = \{6.5, 1.0, 1.1\} \quad (41)$$

The coordinates of $C_1(\bar{x}, \bar{y})$ are $v_2 = \{0.41, 0.041\}$. Eigenvalues are obtained as $\lambda_1 = \{0.018595 + 0.191949 i, \lambda_2 = 0.018595 - 0.191949 i\}$. Hence $C_1(\bar{x}, \bar{y})$ is an unstable spiral point. For an initial point $\{0.545, 0.0545\}$ very close to $C_1(\bar{x}, \bar{y})$, the solution is an enlarging spiral as seen in Fig.10.

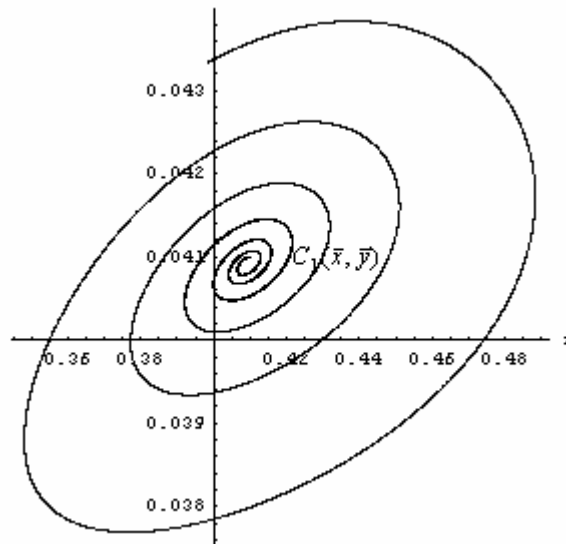


Fig.10. For initial data $v_2 = \{0.41, 0.041\}$ and $\mu_2 = \{6.5, 1.0, 1.1\}$, the solution is an enlarging spiral.

5. CONCLUSION

In this study, a ratio-dependent food chain model is analyzed and possible dynamical behavior of this system investigated at equilibrium points. It has been shown that, the solutions possess Hopf bifurcations. The system has periodic solutions in a small neighborhood of centers. For unstable nodes, for suitable initial conditions, it is seen that the system undergoes limit cycles. In the case of vanishing top predator, the equilibrium points, the stability of solutions, the existence of limit cycles, the Hopf bifurcation and stability of the periodic solution created by the bifurcation are all studied. A limit cycle in the two dimensional stable manifold is seen to be a periodic solution in the three dimensional system.

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